Int. J. Heat Mass Transfer. Vol. 19, pp. 951–953. Pergamon Press 1976. Printed in Great Britain

SHORTER COMMUNICATIONS

HEAT-TRANSFER COEFFICIENTS FOR COMBINED FORCED AND FREE CONVECTION FLOW ABOUT A SEMI-INFINITE, ISOTHERMAL PLATE

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(*Received* 9 *September* 1975 *and in revised form* 21 *January* 1976)

NOMENCLATURE

F, local heat-transfer coefficient;
g, acceleration due to gravity;
Gr, Grashof number, $\dfrac{g\beta\Delta T L^3}{\nu^2}$;
L, characteristic length;
Nu, Nusselt number, $\dfrac{qL}{k\Delta T}$;
Pr, Prandtl number, $\dfrac{\nu}{\kappa}$;
q, local heat transfer per unit time per unit area;
Q, heat-transfer coefficient of Merkin's numerical results;
Re, Reynolds number, $\dfrac{UL}{\nu}$;
T_0, wall temperature;
T_∞, stream temperature;
ΔT, $= T_0 - T_\infty$;
$u, \bar{u}, v, \bar{v}$, dimensionless velocities;
x, dimensionless distance from leading edge;
$y, \bar{y}$, dimensionless distances normal to plate.

Greek symbols

β, coefficient of thermal expansion;
ν, kinematic viscosity;
θ, dimensionless temperature;
ξ, characteristic coordinate, $\dfrac{g\beta\Delta T x}{U^2}$;
η, similarity variable.

Subscripts

L, S, Lloyd and Sparrow;
M, Merkin.

INTRODUCTION

AN EXAMINATION of the series of theoretical and experimental papers [1–8], investigating the flow of a uniform stream U past a semi-infinite vertical plate held at constant temperature T_0, suggests that for general Prandtl number confidence may be placed in heat-transfer coefficient estimates based either on local similarity analysis or series solution representations. It must be recognised however that both of these theoretical methods rely on re-solving two point boundary value problems for each particular Prandtl number so that each new value must in essence be treated as a new problem. This is also true for any further full numerical solution of the problem and it would appear at first sight that the Merkin [6] solution simply served the purpose of providing validation of the series solutions together with specific details for the case $Pr = 1$. In the work that follows it is to be demonstrated that this need not be so. It is shown that the detailed numerical work in fact provides a universal profile for the local heat-transfer coefficients for a large range of Prandtl numbers. The format in which this profile is presented allows estimates of local heat-transfer coefficients to be obtained straightforwardly through simple scaling with respect to any Prandtl number within the appropriate range.

THE UNIVERSAL PROFILE

The dimensionless form of the laminar boundary-layer equations governing the flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{Gr}{Re^2}\theta + \frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial y^2}. \tag{3}$$

Acrivos [9] has shown that appropriate co-ordinate transformations under the constraint of large Prandtl number are

$$\bar{y} = yPr^{\frac{1}{3}}, \quad \bar{u} = uPr^{\frac{1}{3}}, \quad \bar{v} = vPr^{\frac{2}{3}}$$

revealing the form of the boundary-layer equations (1)–(3) relevant to an inner thermal layer as

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{4}$$

$$\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{Gr}{Re^2 Pr^{\frac{1}{3}}}\theta = \frac{1}{Pr}\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) \tag{5}$$

$$\bar{u}\frac{\partial \theta}{\partial x} + \bar{v}\frac{\partial \theta}{\partial \bar{y}} = \frac{\partial^2 \theta}{\partial \bar{y}^2}. \tag{6}$$

It follows immediately that for $Pr \gg 1$, the problem is characterised by the single dimensionless group $Gr/Re^2Pr^{\frac{1}{3}}$ as opposed to the two original groups Gr/Re^2 and Pr. In particular it must be concluded that in the limit as $Pr \to \infty$ the local heat-transfer coefficient is given by

$$\frac{Nu}{Re^{\frac{1}{2}}Pr^{\frac{1}{3}}} = F\left(x, \frac{Gr}{Re^2 Pr^{\frac{1}{3}}}\right). \tag{7}$$

It is a fact of general experiences that analysis based on the hypothesis of a parameter $\ll 1$ or $\gg 1$ will often provide wholly valid information into the parameter range $O(1)$. If it is conjectured that relationship (7) falls into such a category, the conclusion for $Pr = 1$ would be

$$\frac{Nu}{Re^{\frac{1}{2}}} = F\left(x, \frac{Gr}{Re^2}\right). \tag{8}$$

Note that the particular geometry under consideration lacks a specific characteristic length L and that therefore parameters must be interpreted locally with respect to the distance from the leading edge x. The R.H.S. of (8) is then effectively a formulation $F(\xi)$ where ξ is the now well known characteristic co-ordinate of the problem reflecting local relative importance of buoyancy and inertia forces at various

Table 2

ξ	$Pr = 0.72$ $\left(\frac{Nu}{Re^{1/2}}\right)_{L,S}$	$Pr = 0.72$ $\left(\frac{Nu}{Re^{1/2}}\right)$ from (14)	$Pr = 10$ $\left(\frac{Nu}{Re^{1/2}}\right)_{L,S}$	$Pr = 10$ $\left(\frac{Nu}{Re^{1/2}}\right)$ from (14)	$Pr = 100$ $\left(\frac{Nu}{Re^{1/2}}\right)_{L,S}$	$Pr = 100$ $\left(\frac{Nu}{Re^{1/2}}\right)$ from (14)
0	0.2956	0.2976	0.7281	0.7153	1.572	1.514
0.1	0.3158	0.3259	0.7574	0.7424		
0.4	0.3561	0.3686				
1.0	0.4058	0.4238	0.9212	0.8836	1.826	1.721
2.0	0.4584	0.4778	1.029	0.9847	1.994	1.907
4.0	0.5258	0.5574	1.173	1.121	2.232	2.095
10		0.6870		1.351		2.476
100		1.2922		2.51		4.141

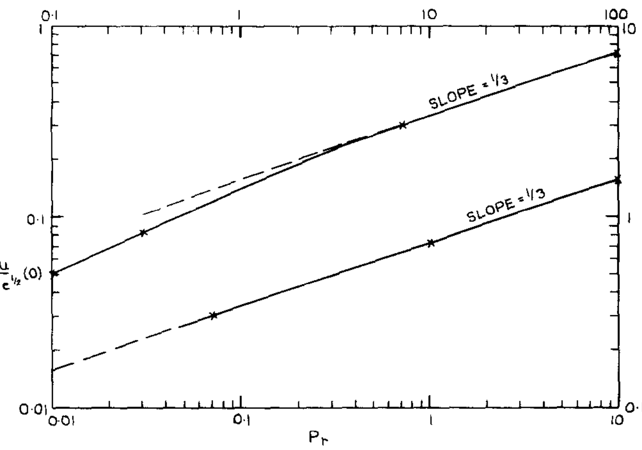

FIG. 1. Leading edge values of heat-transfer coefficient. Upper curve related to left hand ordinate and lower abscissa; Lower curve related to right hand ordinate and upper abscissa: ×, Lloyd and Sparrow [5] computed results.

stations along the plate. Is it possible that

$$\frac{Nu}{Re^{1/2}} = F(\xi) = -\frac{\partial \theta_M}{\partial \xi_M}(\xi, 0) \qquad (9)$$

where M subscript refers to Merkin's results, provides a particular instance of the more general relation

$$\frac{Nu}{Re^{1/2}} = Pr^{1/3} F_M\left(\frac{\xi}{Pr^{1/3}}\right)? \qquad (10)$$

We examine this conjecture in the results.

RESULTS

A preliminary investigation which naturally suggests itself is a comparison of the leading edge estimates of Lloyd and Sparrow's local similarity results with estimates based on (10) namely

$$\frac{Nu}{Re^{1/2}} = Pr^{1/3} F_M(0) = Pr^{1/3}\left\{-\frac{\partial \theta_M}{\partial \eta_M}(0,0)\right\}_{(Pr=1)}. \qquad (11)$$

A detailed examination thus requires a comparison of the forecast

$$\frac{Nu}{Re^{1/2}} = \frac{Pr^{1/3}}{\sqrt{2}} 0.469600 = Pr^{1/3}\, 0.33206. \qquad (12)$$

with the computed values of Lloyd and Sparrow, where the factor $\sqrt{2}$ accounts for a minor difference in formulation between the respective authors. Table 1 demonstrates precisely this comparison.

Table 1

Pr	$\left(\frac{Nu}{Re^{1/2}}\right)_{L,S}$	$\left(\frac{Nu}{Re^{1/2}}\right)$ from (12)
0.03	0.08439	0.1031
0.72	0.2956	0.2976
10	0.7281	0.7153
100	1.571	1.541

These values give cause for optimism since discrepancies for $Pr \geq 0.72$ are less than 2% whilst the larger discrepancy for $Pr = 0.03$ is only to be expected bearing in mind the asymptotic origins of (12). Figure 1 highlights the $Pr^{1/3}$ behaviour for large Prandtl number and also displays the change to $Pr^{1/2}$ behaviour for small Prandtl number in accordance with the expectation that

$$\lim_{Pr \to 0} \frac{Nu}{Re^{1/2}} \sim \sqrt{\left(\frac{Pr}{\pi}\right)}.$$

At general stations along the plate Merkin's results are presented in terms of

$$Q = -\left(\frac{\nu U}{g\beta\Delta T}\right)^{1/2} \frac{1}{\Delta T}\left(\frac{\partial T}{\partial y}\right)_{y=0} = -(2\xi)^{-1/2}\frac{\partial \theta_M}{\partial \eta_M}(\xi, 0) \qquad (13)$$

so that a full examination of the universal profile conjecture requires a direct comparison of

$$\frac{Nu}{Re^{1/2}}(\xi) = Pr^{1/3} Q\left(\frac{\xi}{Pr^{1/3}}\right)\left(\frac{\xi}{Pr^{1/3}}\right)^{1/2} \qquad (14)$$

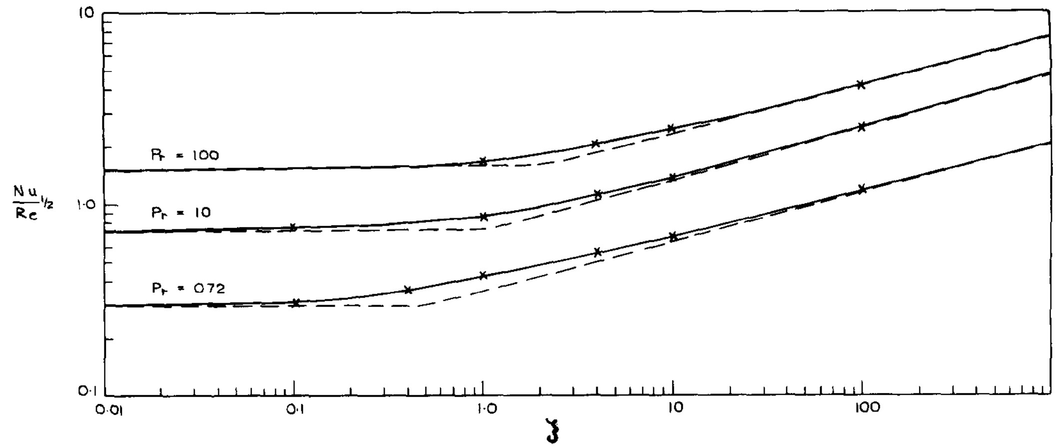

FIG. 2. Local heat-transfer coefficient estimates for forced-free convection flow about an isothermal plate: ×, universal profile values; ---, forced and free convection asymptotes.

with those computed values of Lloyd and Sparrow. Such a comparison is presented in Table 2. Without additional numerical information for Q near $\xi = 0$ further comparison for $Pr = 100$ is difficult because of the rate of change of Q in that vicinity. Otherwise the results of Table 2 are based on simple linear interpolations between Merkin's published results. To facilitate examination of the conjecture with respect to the anticipated free convection asymptotes at large ξ evaluations of (14) at $\xi = 10$, 100 have also been included in Table 2. The level of agreement is always within 7% and in many cases much less than this maximum deviation. Figure 2 illustrates the overall results. Reference to [8] confirms not only a satisfactory comparison with local similarity results, but also a favourable comparison with existing experimental work for the case $Pr = 0.72$.

DISCUSSION

The conjecture that an analysis based upon $Pr \gg 1$ might offer information into the range Pr of $O(1)$ has, to a level of agreement acceptable for practical purposes, been substantiated. The consequence has been to conclude that the numerical solution of Merkin provides a universal profile for the local heat-transfer coefficient when interpreted in accordance with (14). This straightforward scaling of available results might well be used with some degree of confidence for Prandtl numbers greater than 0.4 (see Fig. 1).

Although attention has here been focused on the flat plate geometry with uniform free stream and uniform plate temperature the scaling arguments with respect to large Prandtl number remain true for more general combined free and forced convection flows. It may well be that the conjecture of a universal profile obtained from a particular exact solution holds good in a variety of circumstances over similar ranges of Prandtl number values.

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